From de Finetti's coherence to new theories for reasoning under uncertainty

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1 Extended Abstract

The representation and processing of uncertainty is an active branch of Artificial Intelligence (AI), with many approaches, influenced by different fields like mathematics, computer science, psychology, decision theory, game theory, and economics.

The current pervasiveness of AI has clearly pointed out an urgent need of explainable models, encoding "soft" methods, as well as sound normative theories of uncertainty that implement distinguished agent's behaviors, in a way to achieve trustworthiness of automatic processes.

In this talk we start by presenting some bridges between de Finetti's theory [6, 10] of coherent probability and other more modern non-additive uncertainty theories, complying with conditioning. We refer to de Finetti's celebrated "betting scheme", that forbids all those combinations of bets resulting in a sure win or a sure loss, namely, Dutch books.

The peculiarity of de Finetti's theory consists in the possibility of extending (generally not in a unique way) a coherent probability assessment to any other family of events, by preserving coherence. Furthermore, the set of all the possible coherent extensions gives rise to lower and upper envelopes that reveal to be non-additive uncertainty measures [8]. Actually, the notions of lower and upper probabilities, together with the ensuing notions of lower and upper expectations, have gained a privileged role, departing from probability theory, after the work of Walley [20, 21].

Hence, though de Finetti's betting construction has been originally introduced to justify additive probability measures, it "naturally" gives rise to non-additive uncertainty measures, where lower and upper envelopes encode a pessimistic and an optimistic attitude towards uncertainty, respectively [11, 4]. In this setting we show that, under some specific logical constraints, probabilistic coherence gives rise to distinguished classes of non-additive uncertainty measures and non-linear expectations.

A second important by-product of de Finetti's approach is obtained by generalizing the betting protocol to define coherence directly in a reference non-additive uncertainty calculus. For instance, betting notions of coherence have been directly introduced in Walley's framework, weakening de Finetti's ones.

Here, we consider an analogous approach referring to the so-called α-DS Choquet expectation theory [16] that subsumes many non-additive uncertainty theories, like Dempster-Shafer theory [7, 17], possibility theory [9], and credibility theory [14]. This is particularly relevant in AI, since we characterize the behavior of an agent adopting this uncertainty theory in terms of partial resolving uncertainty (PRU) [13] and the Hurwicz criterion [12].

More in detail, PRU means that, when uncertainty is resolved, the agent may acquire the information that an event has occurred without being able to identify the true state of the world. In this case, the Hurwicz criterion can be seen as the α-mixture between the "best" and the "worst" result of a random payoff on the acquired piece of information, where α is a fixed pessimism index. De Finetti's betting scheme has been generalized to work within the α-DS Choquet expectation theory in [16] and in the subcase of credibility theory in [15].

The attention is then focused on models relying on belief functions in Dempster-Shafer theory, that correspond to $\alpha = 0$. In this setting, the notion of conditioning has a relevant role from both a theoretical and a practical point of view. This is why we present different conditioning rules such as the Bayesian rule [20], the geometric rule [18] and Dempster rules [7], for which a comparison appears in [5, 3].

The adopted conditioning notion is particularly relevant to formulate a theory of imprecise processes (see e.g. [19]). Working in Dempster-Shafer theory, we define a time-homogeneous Markov multiplicative binomial process (DS-multiplicative binomial process), which is characterized by a distinguished family of transition belief functions [2].

A noticeable application of this process is to realize a bid pricing rule in finance [2, 16], defined as a one-step discounted conditional Choquet expectation [1]. An important feature of the quoted imprecise process is its nice parameterization based on two parameters only, that has a direct impact on calibration tasks, favoring scalability.

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